Chapter 1 Systems of Linear Equations

- 1.0 Motivation (動機)
- 1.1 Matrices and Systems of Linear Equations
- 1.2 Gauss-Jordan Elimination
- 1.3 Vector Space $\mathbb{R}^n$
- 1.4 Basis and Dimension
- Business Week: Math Will Rock Your World
- 1.5 Dot Product, Norm, Angle, and Distance
- 1.6 Curve Fitting

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Chinese: first Gaussian elimination (200 BC), Gauss elimination (1800)

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1.0 What is Linear Algebra? (1/2)

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

1. Solve

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

- 雞兔同籠 (自然數): 雞 $x$ 隻，兔 $y$ 隻
  $$x + y = 12$$
  $$2x + 4y = 30$$
- Chapter 1, 2, 3
- Matrix notation (符號): $A x = b, A \in \mathbb{R}^{m \times n}$,
  $x \in \mathbb{R}^{n \times 1}, b \in \mathbb{R}^{m \times 1}$
- Easy to solve
- Nonlinear: $x_1^2 + x_2^3 + \log x_3 = 2$ (difficult to solve)

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What is Linear Algebra? (2/2)

2. Linear mapping between vector spaces

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(x) = A x$

- Chapter 2, 4, 5
- Image rotation (影像旋轉)
  - Rotate the images: 35° and 90°
  - You will learn how these programs works.
  - More detail in the course Image Processing

3. Solve $A x = \lambda x$, where $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n \times 1}, \lambda \in \mathbb{R}, x \neq 0$

- Chapter 3, 5
- det $(A - \lambda I) = 0$, i.e, $(A - \lambda I)$ singular
- Many interesting applications
- These 3 problems are related.

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Why do we learn it? Applications (1/2)

- 矩陣是一種表示東西的數學語言：多變數的二次
  偏導，Digital image, Courses data structure and
  algorithm
- Math: Curve fitting (1), Markov chain (2, 3) (工程與
  商業的應用，3 上選修 OR2), difference and
differential equations (5), Fourier approximation (6)
- Physics: vectors such as force, velocity, …
- Statistics (統計學): least-squares curves (6) Computer
  Science: Cryptography (2), computer graphics (2),
  numerical methods (7), Google’s search engine (3),
  algorithm (5) (演算法，3 下必修)
- Biology: Genetics (2)
Why do we learn it? Applications (2/2)

- **Optimization**: Linear programming in Operations research (8) (2 下必修・OR・作業研究)
  
  Max \( c x \)

  subject to \( A x = b \)

- **Economics**: Leontief IO Model (2) (economics Nobel Prize in 1973)

- **Sociology**: Communication and relationship (2)

- **Electrical engineering**: Electrical networks (1), coding theory (6)

- **Civil engineering**: Traffic flow (1)

- Top 10 algorithms in the 20th century: 3 algorithms from linear algebra (on the course web)

  - Derek Bok (哈佛前校長) 著，大學教了沒？二十一世紀八個教育目標之一：思辨能力

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1.1 Matrices (矩陣) and Systems of Linear Equations (線性方程式系統)

- **Unique solution**
  
  \[
  \begin{align*}
  x + 3y &= 9 \\
  -2x + y &= -4
  \end{align*}
  \]

  Lines intersect (交) at (3, 2). No point of intersection.

  Unique solution, \( x = 3, y = 2. \)

- **No solution**
  
  \[
  \begin{align*}
  -2x + y &= 3 \\
  -4x + 2y &= 2
  \end{align*}
  \]

  Lines are parallel (平行)

  -4x + 2y = 2

  Both equations have the same graph. Any point on the graph is a solution.

---

Definition: A **matrix** is a rectangular array (矩形陣列) of numbers.

The numbers in the array are called the **elements** (元素) of the matrix.

★ **Rows (列) and Columns (行)**

\[
\begin{bmatrix}
2 & 3 & -4 \\
7 & 5 & -1
\end{bmatrix}
\]

row 1

\[
\begin{bmatrix}
2 & 3 & -4 \\
7 & 5 & -1
\end{bmatrix}
\]

row 2

\[
\begin{bmatrix}
2 & 3 & -4 \\
7 & 5 & -1
\end{bmatrix}
\]

column 1

\[
\begin{bmatrix}
2 & 3 & -4 \\
7 & 5 & -1
\end{bmatrix}
\]

column 2

\[
\begin{bmatrix}
2 & 3 & -4 \\
7 & 5 & -1
\end{bmatrix}
\]

column 3

★ **Submatrix (子矩陣)**

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
5 & 1 & -2
\end{bmatrix} \quad P = \begin{bmatrix}
1 & 2 & 3 \\
5 & 1 & -2
\end{bmatrix} \quad Q = \begin{bmatrix}
7 & 3 \\
1 & -2
\end{bmatrix} \quad R = \begin{bmatrix}
1 & 2 & 3 \\
5 & 1 & -2
\end{bmatrix}
\]

matrix A

submatrices of A

★ **Size (大小) and Type (種類)**

\[
\begin{bmatrix}
1 & 0 & 3 \\
-2 & 4 & 5
\end{bmatrix} \quad \begin{bmatrix}
2 & 5 & 7 \\
-9 & 0 & 1
\end{bmatrix} \quad \begin{bmatrix}
4 & -3 & 8 \\
3 & 2
\end{bmatrix}
\]

2×3 matrix

3×3 matrix

1×4 matrix

3×1 matrix

★ **Location (位置)**

\[
\begin{bmatrix}
2 & 3 & -4 \\
7 & 5 & -1
\end{bmatrix}
\]

The element 7 is in row 2, column 1.

location (2, 1).

★ **Identity (單位) Matrices**

\[
I_2 = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \quad I_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Matrix of coefficient, Augmented matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 3 & 1 & 2 \\
1 & -1 & -2 & -6
\end{bmatrix}
\]

matrix of coefficient  

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Solution (2,0,3)

\[
\begin{bmatrix}
1 & 1 & 1 & 2 \\
2 & 3 & 1 & 3 \\
1 & -1 & -2 & -6
\end{bmatrix}
\]

augmented (擴充) matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Solution (t,2-t,0)

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

No solution or (2-s,s,0)

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_1 - x_2 - 2x_3 = -6 \\
x_1 + x_2 = 2
\end{bmatrix}
\]

Example 1

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_1 + 3x_2 + x_3 = 3 \\
x_1 - x_2 - 2x_3 = -6
\end{bmatrix}
\]

Solution

**Equation Method**

Initial system:

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_1 + 3x_2 + x_3 = 3 \\
x_1 - x_2 - 2x_3 = -6
\end{bmatrix}
\]

**Analogous (類) Matrix Method**

Augmented matrix:

\[
\begin{bmatrix}
1 & 1 & 1 & 2 \\
2 & 3 & 1 & 3 \\
1 & -1 & -2 & -6
\end{bmatrix}
\]

Eq2+(–2)Eq1

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_2 - x_3 = -1 \\
-2x_2 - 3x_3 = -8
\end{bmatrix}
\]

R2+(–2)R1

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & -1 \\
0 & -2 & -3 & -8
\end{bmatrix}
\]

Eq3+(–1)Eq1

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The solution is

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_1 - x_2 - 2x_3 = -6 \\
x_1 + x_2 = 2
\end{bmatrix}
\]

R3+(–1)R1

\[
\begin{bmatrix}
1 & 1 & 1 & 2 \\
0 & 1 & -1 & -1 \\
0 & 2 & 3 & 8
\end{bmatrix}
\]

Elementary Transformations (基本轉換)

1. Interchange (交換) two equations.

Elementary Row Operations (基本列運算)

1. Interchange two rows of a matrix.

2. Multiply the elements of a row by a nonzero constant.

3. Add a multiple of one equation to another equation.

The solution is

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_1 + 3x_2 + x_3 = 3 \\
x_1 - x_2 - 2x_3 = -6
\end{bmatrix}
\]

The solution is

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_2 - x_3 = -1 \\
-2x_2 - 3x_3 = -8
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & x_1 + x_2 + x_3 = 2 \\
x_1 - x_2 - 2x_3 = -6 \\
x_1 + x_2 = 2
\end{bmatrix}
\]

2R1 – R2

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -1 & -1 \\
0 & 0 & -5 & -10
\end{bmatrix}
\]

R1+(–1)R2

\[
\begin{bmatrix}
1 & 0 & 2 & 3 \\
0 & 1 & -1 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

R3+(2)R2

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

R3+(2)R2

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & -1 & -1 \\
0 & 2 & 3 & 8
\end{bmatrix}
\]

The solution is

\[
\begin{bmatrix}
x_1 = -1 \\
x_2 = 1 \\
x_3 = 2
\end{bmatrix}
\]

The solution is

\[
\begin{bmatrix}
x_1 = -1 \\
x_2 = 1 \\
x_3 = 2
\end{bmatrix}
\]
Example 4 (Different from high school)

Solving the following three systems of linear equations, all of which have the same matrix of coefficients.

\[
\begin{align*}
\begin{cases}
-x_i - x_2 + 3x_3 &= b_1 \\
2x_i - 4x_2 + x_3 &= b_2 \\
-x_i + 2x_2 - 4x_3 &= b_3
\end{cases}
\end{align*}
\]

for

\[
\begin{align*}
\begin{cases}
b_1 &= 8 \\
b_2 &= 11 \\
b_3 &= -11
\end{cases}
\end{align*}
\]

in turn

\[
\begin{align*}
b_1 &= 2 \\
b_2 &= 3 \\
b_3 &= 6
\end{align*}
\]

\[
\begin{align*}
b_1 &= 3 \\
b_2 &= 2 \\
b_3 &= 1
\end{align*}
\]

**Solution**

\[
\begin{bmatrix}
1 & -1 & 3 & 8 & 0 & 3 \\
2 & -1 & 4 & 11 & 1 & 3 \\
-1 & 2 & -2 & -11 & 2 & -4
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -1 & 3 & 8 & 0 & 3 \\
0 & 1 & -2 & -5 & 1 & -3 \\
0 & 1 & -1 & -3 & 2 & -1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 3 & 1 & 0 \\
0 & 1 & -2 & -5 & 1 & -3 \\
0 & 0 & 1 & 2 & 1 & 2
\end{bmatrix}
\]

The solutions are \( x_1 = 1, x_2 = -1, x_3 = 2 \)

\( x_1 = 0, x_2 = 3, x_3 = 1 \)

\( x_1 = -2, x_2 = 1, x_3 = 2 \).

Example: Unknown Parameter (1/2)

\[
\begin{align*}
x_i - 2x_2 + 3x_3 &= 1 \\
2x_i + kx_2 + 6x_3 &= 6 \\
x_i + 3x_2 + (k - 3)x_3 &= 0
\end{align*}
\]

\[
\begin{align*}
1 & -2 & 3 & 1 \\
2 & k & 6 & 6 \\
-1 & 3 & k - 3 & 0
\end{align*}
\]

\[
\begin{align*}
0 & 1 & k & 1 \\
0 & k + 4 & 0 & 4 \\
0 & 0 & -k(k + 4) & -k
\end{align*}
\]

- Consider \( k (k + 4) \):
  - If \( k = -4, 0 = 4 \Rightarrow \text{no solution.} \)
  - If \( k = 0 \), then infinitely many solutions \((3 - 3t, 1, t)\).

Example: Unknown Parameter (2/2)

- If \( k \neq -4 \) and \( k \neq 0 \), then

\[
\begin{bmatrix}
1 & 0 & 2k + 3 & 3 \\
0 & 1 & k & 1 \\
0 & 0 & -k(k + 4) & -k
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 2k + 3 & 3 \\
0 & 1 & k & 1 \\
0 & 0 & 1 & k + 4
\end{bmatrix}
\]

unique solution

\[
\begin{bmatrix}
k + 9 & 4 & 1 \\
k + 4 & k + 4 & k + 4
\end{bmatrix}
\]

1-2 Gauss-Jordan Elimination (消去法)

**Definition**: A matrix is in reduced 
(簡約) echelon (梯形) form (型式) if

1. Any rows consisting entirely of zeros are grouped at the bottom (底部) of the matrix.
2. The first nonzero element of each row is 1. This element is called a leading (首項) 1.
3. The leading 1 of each row after the first is positioned (置) to the right (右) of the leading 1 of the previous (先前) row.
4. All other elements in a column that contains (包含) a leading 1 are zero.

- Will also be used in section 2.4, 3.3, 4.8
- Suitable for software implementation
In Reduced Echelon Form

\[
\begin{bmatrix}
1 & 0 & 0 & 8 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 9 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Not in Reduced Echelon Form

\[
\begin{bmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 7 & 0 & 8 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

Example 4
\[x_1 + 2x_2 - x_3 + 3x_4 + x_5 = 2 \]
\[2x_1 + 4x_2 - 2x_3 + 6x_4 + 3x_5 = 6 \]
\[-x_1 - 2x_2 + x_3 - 4x_4 + 3x_5 = 4 \]

Solution
\[
\begin{bmatrix}
1 & 2 & -1 & -3 & 3 & 1 & 2 \\
2 & 4 & -2 & 6 & 3 & 6 & 4 \\
-1 & -2 & 1 & -1 & 3 & 4 \\
R_2 - (\cdot) R_1 \\
R_3 - (\cdot) R_1
\end{bmatrix}
\approx
\begin{bmatrix}
1 & 2 & -1 & -3 & 3 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 4 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 \\
R_2 + (\cdot) R_3 \\
R_3 + (\cdot) R_1
\end{bmatrix}
\approx
\begin{bmatrix}
1 & 2 & -1 & 3 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 \\
R_2 (\cdot) R_3 \\
R_3 + (\cdot) R_1
\end{bmatrix}
\]
\[
\begin{bmatrix}
x_1 = -2x_2 + x_3 + 3 \\
x_2 = -1 \\
x_3 = 2
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
x_2 = r, x_3 = s, \quad \text{for any } r \text{ and } s, \\
x_1 = 2r + s + 3, \\
x_3 = 1, x_5 = 2
\end{bmatrix}
\]

Example 5
\[x_1 - x_2 + 2x_3 = 3 \]
\[2x_1 - 2x_2 + 5x_3 = 4 \]
\[x_1 + 2x_2 - x_3 = -3 \]
\[2x_2 + 2x_3 = 1 \]

Solution
\[
\begin{bmatrix}
1 & -1 & 2 & 3 \\
2 & -2 & 5 & 4 \\
1 & 2 & -1 & -3 \\
0 & 2 & 2 & 1
\end{bmatrix}
\approx
\begin{bmatrix}
1 & -1 & 2 & 3 \\
0 & 0 & 1 & -2 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Homogeneous (齊次) Systems of Linear Equations

- Homogeneous: All the terms (項) on the right-hand-side are zeros
  \[x_1 + 2x_2 - 5x_3 = 0 \]
  \[-2x_1 - 3x_2 + 6x_3 = 0 \]
  - Trivial (顯而易見) solution (0, 0, 0)

- Reduced echelon form:
  \[x_1 + 2x_2 - 5x_3 = 0 \quad \Rightarrow \quad \begin{cases} x_1 = 3x_2 = 0 \\ -2x_1 - 3x_2 + 6x_3 = 0 \end{cases} \]

  - Solutions: (-3, 4, r). For example, if r = -1, then
    \[(3, -4, -1) \]
    - Trivial solution by letting r = 0.
1.3 Vector Space (向量空間) $\mathbb{R}^n$

**Definition:** Let $(u_1, u_2, \ldots, u_n)$ be a sequence (數列) of $n$ real numbers (實數). The set of all such sequences is called $n$-space and is denoted $\mathbb{R}^n$. $u_1$ is the first component (元素), $u_2$ is the second component and so on.

- **Definition** $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$
  - $u$ and $v$ are equal (相等) if $u_1 = v_1$, $u_n = v_n$
  - Let $c$ be a scalar
  - **Addition (加法):** $u + v = (u_1 + v_1, \ldots, u_n + v_n)$
  - **Scalar multiplication (純量乘積):** $c u = (c u_1, \ldots, c u_n)$

**Example 4**

Consider the vectors $(4, 1)$ and $(2, 3)$,

$\Rightarrow (4, 1) + (2, 3) = (6, 4)$.

(diagonal 對角 of the parallelogram 平行四邊形)

**Direction (方向) of Scalar Multiplication**

- $c > 1$  
- $0 < c < 1$  
- $-1 < c < 0$  
- $c < -1$

**Theorem 1.3**

Let $u$, $v$, and $w$ be vectors in $\mathbb{R}^n$ and let $c$ and $d$ be scalars.

(a) $u + v = v + u$
(b) $u + (v + w) = (u + v) + w$
(c) $u + 0 = 0 + u = u$
(d) $u + (-u) = 0$
(e) $c(u + v) = cu + cv$
(f) $(c + d)u = cu + d u$
(g) $c(du) = (cd) u$
(h) $1u = u$

**Proof:**

(a) $u + v = (u_1 + v_1, \ldots, u_n + v_n)$
(b) $v + u = (v_1 + u_1, \ldots, v_n + u_n)$

**Linear Combinations (線性組合) of Vectors**

- A linear combination of the vectors $u$, $v$, and $w$:
  
  $a u + b v + c w$

- **Example 5:** $u = (2, 5, -3)$, $v = (-4, 1, 9)$, and $w = (4, 0, 2)$. Determine the linear combination $2 u -3 v + w$

  $= 2(2, 5, -3) -3(-4, 1, 9) + (4, 0, 2)$
  
  $= (4, 10, -6) + (12, -3, -27) + (4, 0, 2)$
  
  $= (4 + 12 + 4, 10 - 3 + 0, -6 - 27 + 2) = (20, 7, -31)$
Subspaces (子空間) of $\mathbb{R}^n$

- Example 6: Consider the subset $W$ of $\mathbb{R}^2$ of vectors of the form $(a, 2a)$
  - 舉例：$(1, 2) \in W, (2, 4) \in W$.
  - $\Rightarrow (1, 2) + (2, 4) = (3, 6) \in W$
  - and $4 \cdot (1, 2) = (4, 8) \in W$

- Proof (證明) of a subspace for arbitrary vectors: Let $u = (a, 2a)$ and $v = (b, 2b)$ be vectors in $W$ and $k$ be a scalar. Then
  \[
  u + v = (a, 2a) + (b, 2b) = (a + b, 2(a + b)) \in W
  \]
  \[
  k \cdot u = k \cdot (a, 2a) = (ka, 2ka) \in W
  \]

Homogeneous Systems of Linear Equations

- Example 7:
  - \[
  x - y + 3z = 0
  \]
  - \[
  y - 5z = 0
  \]
  - \[
  2x - y + z = 0
  \]

- Solutions $(2 r, 5 r, r)$

- The set of solutions $W$ is a subspace:
  \[
  u + v = (2r, 5r, r) + (2s, 5s, s) = (2(r + s), 5(r + s), (r + s)) \in W
  \]
  \[
  k \cdot u = k \cdot (2r, 5r, r) = (2kr, 5kr, kr) \in W
  \]

Proof and Example

- 每個人都是女生
  - 證明：每一個人
  - 推翻此論述：第幾排第幾位是男生
- 證明 (proof) 與舉例說明 (illustration)
  - $2 \times 2^n > n^2 + 3$ for all $n$?
  - 舉例說明：(1) $n = 2, 2 \times 2 > 4 + 3$ (2) $n = 3, 2 \times 8 > 9 + 3$ (3) $n = 4, 2 \times 16 > 16 + 3$ (4) $n = 5, 2 \times 32 > 25 + 3$
  - In fact, $n = 1, 2 \times 1 < 1 + 3$

- Not a subspace: $(a, 1)$
  - $(a, 1) + (b, 1) = (a + b, 2)$
  - $3 \cdot (a, 1) = (3a, 3)$

1.4 Basis (基底) and Dimension (維度)

- Standard (標準) basis: Consider the vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ in $\mathbb{R}^3$. Two properties:
  - (1) Span (展開) $\mathbb{R}^3$: Every vector can be expressed in terms of (以) these three vectors $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$
  - (2) Linear independence (獨立): Consider $p(1, 0, 0) + q(0, 1, 0) + r(0, 0, 1) = 0$ for scalars $p, q,$ and $r$.
    - $p(1, 0, 0) + q(0, 1, 0) + r(0, 0, 1) = (p, q, r) = 0$
      - $\Rightarrow$ unique solution $p = q = r = 0$, called Linear independence
    - $1(0, 0, 0) + 0(1, 1, 0) = 0, p = 1, q = 0$ 係數不全為0，稱線性相依 (linear dependent)

- The number of vectors in a basis is called the dimension of the space.
Example

• Linearly independent and spanning set are DIFFERENT concepts (不同的概念)
  — (1, 0, 0) and (0, 1, 0) linearly independent (li), but not spanning set (ss)
    • li: p (1, 0, 0) + q (0, 1, 0) = (0, 0, 0) ⇒ p = q = 0
    • Not ss: Consider (1, 1, 1) = p (1, 0, 0) + q (0, 1, 0)?
  — (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0) spanning set, but not linearly independent
    • ss: (a, b, c) = a (1, 0, 0) + b (0, 1, 0) + c (0, 0, 1) + 0 (1, 1, 0)
    • Not li: 1 (1, 0, 0) + 1 (0, 1, 0) + 0 (0, 0, 1) + (1) (1, 1, 0) = 0

Example 1

Consider the subset W of $\mathbb{R}^3$ consisting of vectors of the form $(a, b, a+b)$.
• It is a subspace of $\mathbb{R}^3$
• Dimension 2:
  — Span: $(a, b, a+b) = (a, 0, a) + (0, b, b) = a (1, 0, 1) + b (0, 1,1)$
    — Linearly independent:
      $p (1, 0, 1) + q (0, 1, 1) = (0, 0, 0), p (1, 0, 1) + q (0, 1, 1) = (p, q, p + q) \Rightarrow p = q = 0$

Example 3

$x_1 - x_2 + x_3 + 2x_4 = 0$
$x_1 - 3x_2 + 2x_4 = 0$
$2x_1 - x_2 - 2x_3 + 4x_4 = 0$

• Solutions: $(3r - 2s, 4r, r, s)$
• It is a subspace of $\mathbb{R}^4$
• Dimension 2:
  — Span: $(3r - 2s, 4r, r, s) = r (3, 4, 1, 0) + s (-2, 0, 0, 1)$
    — Linearly independent:
      $p (3, 4, 1, 0) + q (-2, 0, 0, 1) = (0, 0, 0, 0),
      p (3, 4, 1, 0) + q (-2, 0, 0, 1) = (3p - 2q, 4p, p, q)
      \Rightarrow p = q = 0$

1.5 Dot Product (點積), Norm (長度), Angle (角度), and Distance (距離)

• Definition:
  Let $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ be two vectors in $\mathbb{R}^n$.
  The dot product of $u$ and $v$:
  \[ u \cdot v \equiv u_1 v_1 + \cdots + u_n v_n \in \mathbb{R} \]

Example 1

$u = (1, -2, 4)$ and $v = (3, 0, 2)$

Solution

$u \cdot v = (1 \times 3) + (-2 \times 0) + (4 \times 2) = 3 + 0 + 8 = 11$
Properties (性質) of the Dot Product

Let \( u, v, \) and \( w \) be vectors in \( \mathbb{R}^n \) and let \( c \) be a scalar. Then

1. \( u \cdot v = v \cdot u \)
2. \( (u + v) \cdot w = u \cdot w + v \cdot w \)
3. \( c(u \cdot v) = u \cdot cv \)
4. \( u \cdot u \geq 0, \) and \( u \cdot u = 0 \) if and only if \( u = 0. \)

**Proof**
1. Let \( u = (u_1, u_2, \ldots, u_n) \) and \( v = (v_1, v_2, \ldots, v_n) \). We get
   
   \( u \cdot v = u_1v_1 + \cdots + u_nv_n \)
   
   by the commutative property of real numbers

2. \( u \cdot u = u_1^2 + \cdots + u_n^2 \geq 0. \)

3. \( u \cdot u = 0 \) if and only if \( u = 0. \)

Norm (長度) of a Vector in \( \mathbb{R}^n \)

The length of this vector is

\( \sqrt{u_1^2 + u_2^2} \)

**Definition**

The norm (length or magnitude) of a vector \( u = (u_1, \ldots, u_n) \) in \( \mathbb{R}^n \) is defined by

\[ \|u\| = \sqrt{u_1^2 + \cdots + u_n^2} \]

**Example 2**

Find the norm of the vectors \( u = (1, 3, 5) \) of \( \mathbb{R}^3 \) and \( v = (3, 0, 1, 4) \) of \( \mathbb{R}^4 \).

**Solution**

\[ |u| = \sqrt{(1)^2 + (3)^2 + (5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35} \]

\[ |v| = \sqrt{(3)^2 + (0)^2 + (1)^2 + (4)^2} = \sqrt{9 + 0 + 1 + 16} = \sqrt{26} \]

**Definition:** normalizing (正規化)

A unit (單位) vector is a vector whose norm is 1.

If \( v \) is a nonzero vector, then the vector

\[ u = \frac{1}{\|v\|} v \]

is a unit vector in the direction (方向) of \( v \).

**Example 3**

(a) Show that the vector \((1, 0)\) is a unit vector.

(b) Find the norm of the vector \((2, -1, 3)\). Normalize this vector.

**Solution**

(a) Thus \((1, 0)\) is a unit vector. \( \| (1, 0) \| = \sqrt{1^2 + 0^2} = 1. \)

(b) \( \| (2, -1, 3) \| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14} \).

The normalized vector is

\[ \frac{1}{\sqrt{14}} (2, -1, 3) = \left( \frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) \]

- This vector is a unit vector: \( \sqrt{\frac{4 + 1 + 9}{14}} = 1 \)
- in the direction of \((2, -1, 3)\).
Theorem

The Cauchy-Schwartz Inequality.
If \( u \) and \( v \) are vectors in \( \mathbb{R}^n \), then
\[
|u \cdot v| \leq |u| |v|
\]
Here \( |u \cdot v| \) denotes the absolute value (絕對值) of the number \( u \cdot v \).

\[
|u_1v_1 + \ldots + u_nv_n| \leq \sqrt{|u_1|^2 + \ldots + |u_n|^2} \sqrt{|v_1|^2 + \ldots + |v_n|^2}
\]

Angle (角度) between 2 nonzero vectors

- \( \cos \theta = \frac{u \cdot v}{|u||v|} \)
- \( u \cdot v = 0 \iff \theta = 90^0 \)

Orthogonal Vectors

- **Definition**: Two nonzero vectors are orthogonal (正交) if the angle between them is a right angle (直角).
- **Theorem 1.4**: Two nonzero vectors \( u \) and \( v \) are orthogonal if and only if \( u \cdot v = 0 \).

**Example**

(a) \((1, 0)\) and \((1, 1)\).
(b) \((2, -3, 1)\) and \((1, 2, 4)\).

**Solution**

(a) \( \cos \theta = (1, 0) \cdot (1, 1) / \sqrt{2} = 1 / \sqrt{2} \Rightarrow \theta = 45^0 \)
(b) \( (2, -3, 1) \cdot (1, 2, 4) = (2 \times 1) + (-3 \times 2) + (1 \times 4) = 2 - 6 + 4 = 0 \). Orthogonal.

Properties of Standard Basis of \( \mathbb{R}^n \)

- Consider the standard basis \( \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \) of \( \mathbb{R}^3 \).
- Observe that \( \|(1,0,0)\| = \sqrt{1^2 + 0^2 + 0^2} \). Unit vector
- Orthogonal:
  - \( (1, 0, 0) \cdot (0, 1, 0) = 0 + 0 + 0 = 0 \)
  - \( (1, 0, 0) \cdot (0, 0, 1) = 0 \)
  - \( (0, 1, 0) \cdot (0, 0, 1) = 0 \)
- Orthonormal (正交) set: A set of unit pairwise (成對) orthogonal vectors

Example

Determine a vector in \( \mathbb{R}^2 \) that is orthogonal to \( (3, -1) \).

**Solution**

- Let the vector \((a, b)\) be orthogonal to \((3, -1)\).
- We get \((a, b) \cdot (3, -1) = 0\)
  \[3a - b = 0\]
  \[b = 3a\]
- Thus any vector of the form \((a, 3a)\) is orthogonal to the vector \((3, -1)\).
- Any vector of this form can be written \(a(1, 3)\). A subspace with dim 1
- The set of all such vectors lie on the line defined by the vector \((1, 3)\).
Theorem 1.6

Let \( u \) and \( v \) be vectors in \( \mathbb{R}^n \).

(a) **Triangle Inequality**

\[ ||u + v|| \leq ||u|| + ||v|| \]

The length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides.

(b) **Pythagorean theorem**

If \( u \cdot v = 0 \), then

\[ ||u + v||^2 = ||u||^2 + ||v||^2. \]

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Distance between Points

Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be two points in \( \mathbb{R}^n \).

The distance between \( x \) and \( y \) is defined by

\[ d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \]

**Example 4**

Determine the distance between the points \( x = (1, -2, 3, 0) \) and \( y = (4, 0, -3, 5) \) in \( \mathbb{R}^4 \).

**Solution**

\[ d(x, y) = \sqrt{(1-4)^2 + (-2-0)^2 + (3-(-3))^2 + (0-5)^2} \]

\[ = \sqrt{9 + 4 + 36 + 25} \]

\[ = \sqrt{74} \]

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Business Week: Math Will Rock Your World (1/3)

- How do you convert written words into math?
  - A single article on Bordeaux wine, for example, turns up in the polytope near France, agriculture, wine, even alcoholism.
  - In each case, Inform's algorithm calculates the relevance of one article to the next by measuring the angle between the two lines.

- 生產與作業管理，供應鏈管理 (applications at IBM)
  - IBM: IBM PC ⇒ 軟體 服務導向
  - … constructed a mathematical model of the company's supply chain. It featured raw materials, trucking schedules, and manufacturing plants.
  - Once the company had a working model, it put it through a mathematical analysis called optimization. The results suggested specific improvements, and the rejiggering sped up IBM's operations and cut costs.

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Business Week: Math Will Rock Your World (2/3)

- 人力資源 (applications at IBM)
  - Professionals are divided into 200 categories.
  - A survey of company e-mail could highlight communication links between employees and the informal social networks that they create. Workers who e-mail each other a lot are more likely to work well together.
  - Calendar data could show which consultants have more free time.
  - When a contract comes through for, say, a new call center in Manila, IBM's optimization program will cull through its global database and put together the perfect team.
Business Week: Math Will Rock Your World (3/3)

- 行銷 (Applications at Harrah's Entertainment Inc.)
  - The models include gamblers’ ages, gender, and Zip codes, as well as the amount of time they spent gambling and how much they won or lost. (統計)
  - Study gambling through a host of variables and to target individuals with offers, from getaway weekends to gourmet dining, calculated to maximize returns.
  - In the last five years, Harrah’s has averaged 22% annual growth, and its stock has nearly tripled.
- 財務
  - A generation ago, quants turned finance upside down.
  - Now they’re mapping out ad campaigns and building new businesses from mountains of personal data
- 資管：資訊 + 管理 + 數學 (+ 英文閱讀)

1-6 Curve Fitting (曲線吻合)

- A set of data points \((x_i, y_i), (x_2, y_2), \ldots, (x_n, y_n)\) is given
- The \(x\)-coordinates \((x_i)\) are called base points (基點)
- Goal: Find a polynomial are called base points (基點) whose graph passes through (經過) the points.
- 應用：電腦造字
- It can be shown (later) that if the base points are all distinct (不同), then a unique polynomial of degree (次數) \(n-1\) (or less)
  \[ y = a_0 + a_1 x + \ldots + a_{n-2} x^{n-2} + a_{n-1} x^{n-1} \]
  can be fitted to the points.
  (n 個點，n 個變數)

Example 1

Determine the equation of the polynomial of degree 2 whose graph passes through the points \((1, 6), (2, 3), (3, 2)\).

Solution

The polynomial \(y = a_0 + a_1 x + a_2 x^2\)

\((1) x = 1, y = 6; (2) x = 2, y = 3; (3) x = 3, y = 2\), we have

\[ a_0 + a_1 + a_2 = 6 \]
\[ a_0 + 2a_1 + 4a_2 = 3 \]
\[ a_0 + 3a_1 + 9a_2 = 2 \]

Solving this system and get \(a_0 = 11, a_1 = -6, a_2 = 1\). The parabola (拋物線) is \(y = 11 - 6x + x^2\).