Chapter 8 Calculus of Several Variables

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8.1 Functions of Several Variables

- A real-valued function of two variables, $f$, consists of
  1. A set $A$ of ordered pairs of real numbers $(x, y)$ called the domain of the function
  2. A rule that associates with each ordered pair in the domain of $f$ one and only one real number, denoted by $z = f(x, y)$

- Independent variables $x$ and $y$, dependent variable $z$

- Example 1: Function $f(x, y) = x + xy + y^2 + 2$.

  $f(1, 2) = 1 + (1)(2) + 2^2 + 2 = 9$, $f(2, 1) = 2 + (2)(1) + 1^2 + 2 = 7$
Example 2: Find the domain of each of the following functions:

- (a) \( f(x, y) = x^2 + y^2 \). All points in the \( xy \)-plane
- (b) \( g(x, y) = \frac{2}{x-y} \). Defined for all \( x \neq y \).
- (c) \( h(x, y) = \sqrt{1-x^2-y^2} \). Require \( 1-x^2-y^2 \geq 0 \)

Example 3: Demand equations, \( x \) and \( y \), for the assembled and kit versions of the loudspeaker systems in terms of the unit prices \( p \) and \( q \) are

\[
p = 300 - \frac{1}{4}x - \frac{1}{8}y, \quad q = 240 - \frac{1}{8}x - \frac{3}{8}y
\]

- Weekly total revenue function

\[
R(x, y) = xp + yq
\]
\[= x \left( 300 - \frac{1}{4}x - \frac{1}{8}y \right) + y \left( 240 - \frac{1}{8}x - \frac{3}{8}y \right)\]

\[= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y\]

- Domain of the function \( R \) (Figure 8.2)

\[300 - \frac{1}{4}x - \frac{1}{8}y \geq 0, \ 240 - \frac{1}{8}x - \frac{3}{8}y \geq 0, \ x \geq 0, \ y \geq 0\]

- Points in three-dimensional Cartesian coordinate system (Figure 8.4)

- If \( z = f(x, y) \), then the graph \((x, y, z) = (x, y, f(x, y))\) is usually a surface in the three-dimensional space (Figure 8.5)
Level Curves

• Suppose \( f(x, y) \) is a function of two variables \( x \) and \( y \) with a graph in Figure 8.7.

• The equation \( f(x, y) = c \) describes a curve lying on the plane \( z = c \) called the trace of the graph of \( f \) in the plane \( z = c \).

• If this trace is projected onto the \( xy \)-plane, the resulting curve in the \( xy \)-plane is called a level curve.

• Contour map: Draw the level curves corresponding to several admissible values of \( c \).

• Example 5: Sketch a contour map for the function \( f(x, y) = x^2 + y^2 (\geq 0) \). (Figure 8.9)
8.2 Partial Derivatives

• Suppose $f(x, y)$ is a function of the two variables $x$ and $y$. Then, the first partial derivative of $f$ with respect to $x$ at the point $(x, y)$ is

$$
\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
$$

provided the limit exists. The first partial derivative of $f$ with respect to $y$ at the point $(x, y)$ is

$$
\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}
$$

provided the limit exists.

– Other notation: $f_x, f_y$
• **Example 1:** Consider \( f(x, y) = x^2 - xy^2 + y^3 \):

\[
\frac{\partial f}{\partial x} = 2x - y^2, \quad \frac{\partial f}{\partial y} = -2xy + 3y^2
\]

- **Rate of change of the function \( f \) in the \( x \)-direction at the point \((1, 2)\):** Then

\[
f_x(1, 2) = \left. \frac{\partial f}{\partial x} \right|_{(1,2)} = 2(1) - 2^2 = -2.
\]

That is, \( f \) decreases 2 units for each unit increase in the \( x \)-direction while \( y \) is being kept constant \((y = 2)\).

- **Rate of change of the function \( f \) in the \( y \)-direction at the point \((1, 2)\):** Then

\[
f_y(1, 2) = \left. \frac{\partial f}{\partial y} \right|_{(1,2)} = -2(1)(2) + 3(2)^2 = 8.
\]

That is, \( f \) increases 8 units for each unit increase in the
y-direction while $x$ is being kept constant ($x = 1$).

- Example 2: Compute the first partial derivatives

  - (a) $f(x, y) = \frac{xy}{x^2+y^2}$

    \[
    \frac{\partial f}{\partial x} = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}
    \]

    \[
    \frac{\partial f}{\partial y} = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}
    \]

  - (c) $h(u, v) = e^{u^2-v^2}$

    \[
    \frac{\partial h}{\partial u} = e^{u^2-v^2} \cdot 2u = 2ue^{u^2-v^2}
    \]

    \[
    \frac{\partial h}{\partial v} = e^{u^2-v^2} \cdot (-2v) = -2ve^{u^2-v^2}
    \]
\( - \) (d) \( f(x, y) = \ln(x^2 + 2y^2) \)

\[
\frac{\partial f}{\partial x} = \frac{2x}{x^2 + 2y^2} \\
\frac{\partial f}{\partial y} = \frac{4y}{x^2 + 2y^2}
\]

• Example 3: Consider \( f(x, y, z) = xyz - xe^{yz} + x \ln y \)

\[
f_x = yz - e^{yz} + \ln y \\
f_y = xz - xze^{yz} + \frac{x}{y} \\
f_z = xy - xye^{yz}
\]
Cobb-Douglas Production Function

\[ f(x, y) = ax^b y^{1-b} \]

where \( a \) and \( b \) are positive constants with \( 0 < b < 1 \).

- Here, \( x \) stands for the amount of money expended for labor, \( y \) stands for the cost of capital equipment (buildings, machinery, and other tools of production), and the production function \( f \) measures the output of the finished product (in suitable units).

- Marginal productivity of labor: Partial derivative \( f_x \). It measures the rate of change of production with respect to the amount of money expended for labor,
with the level of capital expenditure held constant.

- Marginal productivity of capital: Partial derivative $f_y$

- Example 4: Production function $f(x, y) = 30x^{2/3}y^{1/3}$ of a certain country in the early years following World War II

\[
f_x = 30 \cdot \frac{2}{3} x^{-1/3} y^{1/3} = 20 \left( \frac{y}{x} \right)^{1/3}
\]

\[
f_y = 30 x^{2/3} \cdot \frac{1}{3} y^{-2/3} = 10 \left( \frac{x}{y} \right)^{2/3}
\]

\[
f_x(125, 27) = 20 \left( \frac{27}{125} \right)^{1/3} = 20 \left( \frac{3}{5} \right) = 12
\]
\[ f_y(125, 27) = 10 \left( \frac{125}{27} \right)^{2/3} = 10 \left( \frac{25}{9} \right) = 27 \frac{7}{9} \]

- Should the government have encouraged capital investment rather than increasing expenditure on labor to increase the country’s productivity?

- Answer: Because \( f_y(125, 27) > f_x(125, 27) \), the government should have encouraged increased spending on capital rather than on labor during the early years of reconstruction.
Second-Order Partial Derivatives

\[ f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (f_x), \quad f_{xy} \equiv \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} (f_x) \]

\[ f_{yx} \equiv \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} (f_y), \quad f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (f_y) \]

- Note that, in general, it is not true that \( f_{xy} = f_{yx} \).
- Theorem: If a function has continuous second-order partial derivatives, then \( f_{xy} = f_{yx} \).
- Example 6: \( f(x, y) = x^3 - 3x^2y + 3xy^2 + y^2 \). Then

\[ f_x = \frac{\partial}{\partial x} (x^3 - 3x^2y + 3xy^2 + y^2) = 3x^2 - 6xy + 3y^2 \]
\[
\begin{align*}
f_y &= \frac{\partial}{\partial y}(x^3 - 3x^2 y + 3xy^2 + y^2) = -3x^2 + 6xy + 2y \\
f_{xx} &= \frac{\partial}{\partial x}(f_x) = 6x - 6y \\
f_{xy} &= \frac{\partial}{\partial y}(f_x) = -6x + 6y, \quad f_{yx} = \frac{\partial}{\partial x}(f_y) = -6x + 6y \\
f_{yy} &= \frac{\partial}{\partial y}(f_y) = 6x + 2
\end{align*}
\]

- Continuous second-order partial derivatives

- Example 7: Function \( f(x, y) = e^{xy^2} \). Then
\[
\begin{align*}
f_x &= \frac{\partial}{\partial x}(e^{xy^2}) = y^2 e^{xy^2}, \quad f_y &= \frac{\partial}{\partial y}(e^{xy^2}) = 2xy e^{xy^2}
\end{align*}
\]
\[ f_{xx} = \frac{\partial}{\partial x}(f_x) = y^4 e^{xy^2} \]

\[ f_{xy} = \frac{\partial}{\partial y}(f_x) = 2ye^{xy^2} + (y^2)(2xy)e^{xy^2} = 2ye^{xy^2}(1+xy^2) \]

\[ f_{yx} = \frac{\partial}{\partial x}(f_y) = 2ye^{xy^2} + (2xy)(y^2)e^{xy^2} = 2ye^{xy^2}(1+xy^2) = f_{xy} \]

\[ f_{yy} = \frac{\partial}{\partial y}(f_y) = 2xe^{xy^2} + (2xy)(2xy)e^{xy^2} = 2xe^{xy^2}(1+2xy^2) \]

- Continuous second-order partial derivatives
8.3 Maxima and Minima of Functions of Several Variables

- Relative maximum (or minimum) at \((a, b)\): If \(f(x, y) \leq (\text{or} \geq) f(a, b)\) for all points \((x, y)\) sufficiently close to \((a, b)\). The number \(f(a, b)\) is called a relative maximum (or minimum) value.

- If the inequalities hold for all points \((x, y)\) in the domain of \(f\), then \(f\) has an absolute maximum (or absolute minimum) at \((a, b)\) with absolute maximum value (or absolute minimum value) \(f(a, b)\).

- (Euler, 1755) From Figure 8.17, it is clear that at the point \((a, b)\) the slope of the "tangent lines" to the surface in any direction must be zero. In particular,
\[
\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0
\]

- The point \((a, b)\) is called a critical point of the function \(f\).
- Figure 8.18: We have \(\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0\). But the point \((a, b, f(a, b))\) is neither a relative maximum nor a relative minimum. Such a point is called a saddle point.
- Example:
  \[f(x_1, x_2) = -4x_1 + 2x_1^2 - 6x_1x_2 + 12x_2 + 4x_2^2.\] Then
  \[
  \frac{\partial f}{\partial x_1} = -4 + 4x_1 - 6x_2, \quad \frac{\partial^2 f}{\partial x_1^2} = 4 > 0
  \]
\[
\frac{\partial f}{\partial x_2} = -6x_1 + 12 + 8x_2, \quad \frac{\partial^2 f}{\partial x_2^2} = 8 > 0
\]

\[
\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = -6
\]

- Concave upward in each coordinate direction.

Could we conclude that the critical point \((10, 6)\) is a local minimum? No

\[
\frac{\partial f}{\partial x_1} = 0 \Rightarrow x_1^*(x_2) = \frac{3}{2}x_2 + 1
\]

\[
\Rightarrow f(x_1^*(x_2), x_2) = -2 + 6x_2 - \frac{x_2^2}{2}
\]

- Need to consider mixed second-order partial
derivatives. In this case, it is $-6$.

- Determining relative extrema:
  1. Find the critical points by solving $f_x = f_y = 0$
  2. Second derivative test: Let

\[
D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}.
\]

  - If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(x, y)$ has a relative maximum at the point $(a, b)$
  - If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(x, y)$ has a relative minimum at the point $(a, b)$
  - If $D(a, b) < 0$, then $f(x, y)$ has neither a relative maximum nor a relative minimum at the point
(a, b)

- If \( D(a, b) = 0 \), then the test is inconclusive. Other techniques must be used to solve the problem.

- Note: \( f(a, b) \) relative maximum \( \iff -f(a, b) \) relative minimum.

- Consider \( -f(x, y) \). Let \( D(x, y) = \begin{vmatrix} -f_{xx} & -f_{xy} \\ -f_{yx} & -f_{yy} \end{vmatrix} \).

- Then \( f(a, b) \) relative maximum \( \iff -f(a, b) \) relative minimum \( \iff -f_{xx} > 0 \) and 
\[ (-f_{xx})(-f_{yy}) - (-f_{xy})(-f_{yx}) > 0 \]

- This test is for 2 variables. For more than 2
variables, we could use "eigenvalues" in the course "Linear Algebra."

- Example 1: \( f(x, y) = x^2 + y^2 \)

\[
\begin{align*}
    f_x &= 2x, \\
    f_y &= 2y, \\
    f_{xx} &= f_{yy} = 2, \\
    f_{xy} &= f_{yx} = 0
\end{align*}
\]

- Setting \( f_x = f_y = 0 \), the sole critical point \( (0, 0) \)

- Compute \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 4 \). In particular, \( D(0, 0) = 4 \).

- Since \( D(0, 0) > 0 \) and \( f_{xx}(0, 0) = 2 > 0 \), the function \( f \) has a relative minimum at \( (0, 0) \).

- The absolute minimum because \( x^2 + y^2 \geq 0 \). (Figure 8.19)
• Example: $f(x, y) = x^4 + y^4$ inconclusive at $(0, 0)$. In fact, the absolute minimum because $x^4 + y^4 \geq 0$.

• Example: Absolute maximum of $f(x, y) = -x^2 - y^2$ at $(0, 0)$

• Example: $f(x, y) = -x^2 + y^2$, saddle point at $(0, 0)$

• Example 2: $f(x, y) = 3x^2 - 4xy + 4y^2 - 4x + 8y + 4$
  \[ f_x = 6x - 4y - 4, \quad f_y = -4x + 8y + 8, \quad f_{xx} = 6, \quad f_{yy} = 8, \quad f_{xy} = f_{yx} = -4 \]
  \[ \text{Setting } f_x = f_y = 0, \text{ the sole critical point } (0, -1) \]
  \[ \text{Compute } D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 32. \text{ So } D(0, -1) = 32. \]
  \[ \text{Since } D(0, -1) > 0 \text{ and } f_{xx}(0, -1) = 6 > 0, \text{ the} \]
function $f$ has a relative minimum at $(0, -1)$.

**Example 3:** $f(x, y) = 4y^3 + x^2 - 12y^2 - 36y + 2$

- $f_x = 2x, f_y = 12y^2 - 24y - 36, f_{xx} = 2, f_{yy} = 24y - 24, f_{xy} = f_{yx} = 0$

- Setting $f_x = f_y = 0$, so $(0, -1)$ and $(0, 3)$ are critical points of $f$.

- Then $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 48(y - 1)$

- At $(0, -1)$: Compute $D(0, -1) = -96 < 0$. Saddle point

- At $(0, 3)$: Compute $D(0, 3) = 96 > 0$ and $f_{xx}(0, 3) = 2 > 0$. Relative minimum with value $f(0, 3) = -106$. 
8.4 Method of Least Squares

- Method of least squares: Given a set of data points scattered about a straight line, determine a straight line that minimizes the sum of the square of the errors.

- For example, given five data points \( P_i(x_i, y_i) \) with a scatter diagram (Figure 8.21)

- If we try to fit a straight line to these data points, the errors are the amounts \( d_i \). (Figure 8.22)

- Principle of least squares: Find the straight line \( L = mx + b \) to minimize \( d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 \).

- The line \( L \) is called the least-squares line, or
regression line.

- Procedure: Find $m$ and $b$ to minimize $f(m, b)$, where

\[
f(m, b) = d_1^2 + \cdots + d_5^2 = (mx_1 + b - y_1)^2 + \cdots + (mx_5 + b - y_5)^2
\]

- Partial derivatives:

\[
\frac{\partial f}{\partial m} = 2(mx_1 + b - y_1)x_1 + \cdots + 2(mx_5 + b - y_5)x_5
\]

\[
= 2[(x_1^2 + \cdots + x_5^2)m + (x_1 + \cdots + x_5)b - (x_1y_1 + \cdots + x_5y_5)]
\]
\[
\frac{\partial f}{\partial b} = 2(mx_1 + b - y_1) + \cdots + 2(mx_5 + b - y_5)
\]
\[
= 2[(x_1 + \cdots + x_5)m + 5b - (y_1 + \cdots + y_5)]
\]

- Equations \( \frac{\partial f}{\partial m} = \frac{\partial f}{\partial b} = 0 \) are called the normal equations.

- Example 1: Find an equation of the least-squares line for the data

\( P_1(1, 1), P_2(2, 3), P_3(3, 4), P_4(4, 3), P_5(5, 6) \).

- The following deviations do not need to memorize the above formulas. It is a good way to do, especially during an exam.

- Straight line: \( L = mx + b \)
Minimize

\[ f(m, b) = (1m + b - 1)^2 + (2m + b - 3)^2 + (3m + b - 4)^2 + (4m + b - 3)^2 + (5m + b - 6)^2 \]

\[ = (1 + 4 + 9 + 16 + 25)m^2 + 5b^2 + (1 + 9 + 16 + 9 + 36) + 2(1 + 2 + 3 + 4 + 5)mb \]
\[ -2(1 + 6 + 12 + 12 + 30)m - 2(1 + 3 + 4 + 3 + 6)b \]
\[ = 55m^2 + 5b^2 + 71 + 30mb - 122m - 34b \]

\[ \Rightarrow f_m = 110m + 30b - 122, \quad f_b = 10b + 30m - 34 \]
\[ f_{mm} = 110, \ f_{bb} = 10, \ f_{mb} = f_{bm} = 30 \]

- Setting \( f_m = f_b = 0 \) \( \Rightarrow m = 1, b = 0.4 \)
- Because \( D(1, 0.4) = (110)(10) - 30^2 = 200 > 0 \) and \( f_{mm} > 0 \), the point \( (1, 0.4) \) is a relative minimizer.
Applications

- Example 2: The proprietor of the Leisure Travel Service compiled the following data relating the firm’s annual profit $y$ to its annual advertising expenditure $x$ (both measured in thousands of dollars).

<table>
<thead>
<tr>
<th>$x$</th>
<th>12</th>
<th>14</th>
<th>17</th>
<th>21</th>
<th>26</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>60</td>
<td>70</td>
<td>90</td>
<td>100</td>
<td>100</td>
<td>120</td>
</tr>
</tbody>
</table>

- Figure 8.24 Scatter diagram

- Normal equations:

$$6b + 120m = 540, 120b + 2646m = 11,530$$
\[ \Rightarrow m = 2.97, b = 30.6 \]

- Least-square line: \( y = f(x) = 2.97x + 30.6 \)

- What is the predicted annual profit if the annual advertising budget is $20,000? Answer:
  \[ f(20) = 2.97(20) + 30.6 = 90, \text{ or } $90,000. \]

**Example 3:** Monthly sales \( x \) (in thousands) with a proposed wholesale unit price of \( p \) dollars

<table>
<thead>
<tr>
<th>( p )</th>
<th>38</th>
<th>36</th>
<th>34.5</th>
<th>30</th>
<th>28.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>2.2</td>
<td>5.4</td>
<td>7.0</td>
<td>11.5</td>
<td>14.6</td>
</tr>
</tbody>
</table>

- Find the demand equation if the demand curve is the least-squares line for these data.

* Normal equations
\[5b + 40.7 = 167\]
\[40.7b + 428.41m = 1280.6\]

\[\Rightarrow m \approx -0.81, \ b \approx 39.99\]

* Lease-squares line \( p = f(x) = -0.81x + 39.99 \)
  (required demand function) for \( 0 \leq x \leq 49.37 \)

- If the total monthly cost function is \( C(x) = 4x + 25 \). Determine the unit wholesale price that will maximize company’s monthly profit.
  
* Total revenue function \( R(x) = xp = -0.81x^2 + 39.99x \)
Profit function

\[ P(x) = R(x) - C(x) = -0.81x^2 + 35.99x - 25 \]

Compute \( P'(x) = -1.62x + 35.99 \). Since \( P'(x) = 0 \Rightarrow x \approx 22.22 \) the only critical point of \( P \).

From \( P(0) = -25, P(22.22) = 374.78, P(49.37) = -222.47 \), we see that the optimal wholesale price is

\[ -0.81 \times 22.22 + 39.99 = \$22.22 \]
8.5 Constrained Maxima and Minima and Method of Lagrange Multipliers

- Example 1: Find the relative minimum of the function \( f(x, y) = 2x^2 + y^2 \) subject to the constraint \( g(x, y) = x + y - 1 = 0 \).
  - From the constraint equation, \( y = 1 - x \).
  - Substituting this value of \( y \) into \( f \) results in a function of \( x \),
  \[
  h(x) = 2x^2 + (1-x)^2 = 3x^2 - 2x + 1 \Rightarrow h'(x) = 6x - 2
  \]
  - Setting \( h'(x) = 0 \) gives \( x = \frac{1}{3} \) as the sole critical point of the function \( h \).
Next, $h''(x) = 6$, and, in particular, $h''\left(\frac{1}{3}\right) = 6 > 0$.

By the second derivative test, the point $x = \frac{1}{3}$ gives rise to a relative minimum of $h$. In this case, $y = 1 - \frac{1}{3} = \frac{2}{3}$.

The point $\left(\frac{1}{3}, \frac{2}{3}\right)$ is the constrained relative minimum of $f$ and $f\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{2}{3}$.

It may be shown that $\frac{2}{3}$ is in fact a constrained absolute minimum value of $f$ (Figure 8.26).

For unconstrained optimization of $f$, the absolute minimum of $f$ is $(0, 0)$ with value 0.
Method of Lagrange Multipliers

- Major drawback of the technique used in Example 1: Need to solve the constrained equation \( g(x, y) = 0 \) for \( y \) explicitly in terms of \( x \)

- Method of Lagrange Multipliers (Lagrange, 1797): Find the relative extremum of \( f(x, y) \) subject to \( g(x, y) = 0 \). (First order necessary condition)
  1. Form an auxiliary function

\[
F(x, y, \lambda) = f(x, y) + \lambda g(x, y)
\]

called the Lagrange function. The variable \( \lambda \) is called the Lagrange multiplier.
2. Solve the system that consists of the equations

\[ F_x = 0, \quad F_y = 0, \quad F_\lambda = 0 \]

for all values of \( x, y, \) and \( \lambda \).

3. Evaluate \( f \) at each of the points \((x, y)\) found in step 2. The largest (smallest) of these values is the maximum (minimum) value of \( f \).

• Example 2: Find the relative minimum of the function \( f(x, y) = 2x^2 + y^2 \) subject to the constraint \( x + y = 1 \).

1. Lagrangian function

\[ F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = 2x^2 + y^2 + \lambda(x+y-1) \]
2. Partial derivatives

\[ F_x = 4x + \lambda = 0, \quad F_y = 2y + \lambda = 0, \quad F_\lambda = x + y - 1 = 0 \]

\[ \implies x = \frac{1}{3}, \quad y = \frac{2}{3}, \quad \lambda = -\frac{4}{3} \]

3. Check its neighboring points:
- (1) Consider \((\frac{1}{3} + 0.001, \frac{2}{3} - 0.001)\). Then
  \[ f(\frac{1}{3} + 0.001, \frac{2}{3} - 0.001) \approx 0.66667. \]
- (2) Consider \((\frac{1}{3} - 0.001, \frac{2}{3} + 0.001)\). Then
  \[ f(\frac{1}{3} - 0.001, \frac{2}{3} + 0.001) \approx 0.66667 \]
- (3) Note that \(f(\frac{1}{3}, \frac{2}{3}) \approx 0.66667.\)
- Therefore, \(x = \frac{1}{3}\) and \(y = \frac{2}{3}\) is a constrained minimum of \(f\).
There exists conditions for the optimality by the second derivative test. But it is more involved. You could check out books in nonlinear programming. (Sufficient condition)

Example 3: Find the relative minimum of the function \( f(x, y, z) = 2xy + 6yz + 8xz \) subject to the constraint \( xyz = 12,000 \).

1. Lagrangian function:
   \[
   F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z)
   \]
   \[
   = 2xy + 6yz + 8xz + \lambda(12,000 - 12)
   \]

2. Partial derivatives:
   \[
   F_x = 2y + 8z + \lambda yz = 0, \quad F_y = 2x + 6z + \lambda xz = 0, \quad F_z = 6y + 8x + \lambda xy = 0, \quad F_{\lambda} = xyz - 12,000 = 0
   \]
\[ \lambda = \frac{-2y + 8z}{yz} = \frac{-2x + 6z}{xz} = \frac{-6y + 8x}{xy} \]

Then

\[ \frac{2y + 8z}{yz} = \frac{2x + 6z}{xz} \Rightarrow 2xy + 8xz = 2xy + 6yz \Rightarrow x = \frac{3}{4}y \]

\[ \frac{2x + 6z}{xz} = \frac{6y + 8x}{xy} \Rightarrow 2xy + 6yz = 6yz + 8xz \Rightarrow z = \frac{1}{4}y \]

\[ \Rightarrow xyz = 12,000 \Rightarrow \left(\frac{3}{4}y\right) (y) \left(\frac{1}{4}y\right) = 12,000 \Rightarrow y = 40 \]

3. Constrained minimum point

\[ x = \frac{3}{4}(40) = 30, \ y = 40 \text{ and } z = \frac{1}{4}(40) = 10 \text{ with } \]

value \( f(30, 40, 10) = 7200. \)
Applications

- Example 6: Suppose $x$ units of labor and $y$ units of capital are required to produce
  
  \[ f(x, y) = 100x^{3/4}y^{1/4} \]

  units of a certain product (Cobb-Douglas production function). If each unit of labor costs $200 and each unit of capital costs $300 and a total of $60,000 is available for production, determine how many units of labor and how many units of capital should be used in order to maximize production.

  - Budget constraint
    
    \[ g(x, y) = 200x + 300y - 60,000 = 0 \]

  - Maximize $f(x, y) = 100x^{3/4}y^{1/4}$ subject to
\[ g(x, y) = 0 \]

- Lagrangian function

\[ F(x, y, \lambda) = f(x, y) + \lambda g(x, y) \]
\[ = 100x^{3/4}y^{1/4} + \lambda(200x + 300y - 60,000) \]

- Critical point(s)

\[ F_x = 75x^{-1/4}y^{1/4} + 200\lambda = 0 \]
\[ F_y = 25x^{3/4}y^{-3/4} + 300\lambda = 0 \]
\[ F_\lambda = 200x + 300y - 60,000 = 0 \]

- From the first equation, \( \lambda = -\frac{3}{8} \left( \frac{y}{x} \right)^{1/4} \)

- Substitute it into the second equation, \( x = \frac{9}{2}y \)
– Substitute these into the third equation, \( y = 50 \) (capital)

– Hence, labor \( x = 225 \)

– Lagrange multiplier

\[
\lambda = -\frac{3}{8} \left( \frac{y}{x} \right)^{1/4} = -\frac{3}{8} \left( \frac{50}{225} \right)^{1/4} \approx -0.257
\]

– Marginal productivity of money \((-\lambda)\)

* If one additional dollar is available for production, then approximately \(-\lambda\) units of a product can be produced.

* For example, if $65,000 is available for production instead of the originally budgeted figure of $60,000, then the maximum
production may be boosted from the original
\[ f(225, 50) = 100(225)^{3/4}(50)^{1/4} = 15,448 \text{ units} \]
to \[ 15,448 + 5000(0.257) = 16,733 \text{ units}. \]

* Exact: \( F_x = F_y = 0 \) same as before. Except

\[ F_\lambda = 200x + 300y - 65,000 = 200 \left( \frac{9}{2} y \right) + 300y - 65,000 = 0 \]

\[ \Rightarrow y \approx 54.17 \Rightarrow x \approx 243.75 \]

\[ \Rightarrow f(243.75, 54.17) \approx 16,736 \]

- Example 1 in Section 4.5: Maximize the area \( (xy) \)
  subject to the perimeter constraint \( 2x + 2y = 50 \)
  - Lagrangian function
\[ F(x, y, \lambda) = xy + \lambda(x + y - 25) \]

- Critical point(s)

\[ F_x = y + \lambda = 0, \quad F_y = x + \lambda = 0 \]
\[ F_\lambda = x + y - 25 = 0 \]

\[ \Rightarrow -\lambda = x = y \Rightarrow x = y = 12.5 \]

- Example: Find out the shortest distance from the point \((2, 3)\) to the line \(x + y = 1\).

- Minimize \((2 - x)^2 + (3 - y)^2\) subject to \(x + y = 1\)
- \( F(x, y, \lambda) = (2 - x)^2 + (3 - y)^2 + \lambda(x + y - 1) \)
- \( F_x = -2(2 - x) + \lambda, \quad F_y = -2(3 - y) + \lambda, \quad F_{\lambda} = x + y - 1 \Rightarrow x = 0, y = 1, \lambda = 4 \)

- Geometric interpretation: Pick the other point \((1, 0)\) on the line \(x + y = 1\). Then \((1, 0) - (0, 1) = (1, -1)\) and \((2, 3) - (0, 1) = (2, 2)\). Take the inner product of these 2 vectors

\[
(1, -1) \cdot (2, 2) = 1 \times 2 + (-1) \times 2 = 0 = \sqrt{1^2 + (-1)^2} \sqrt{2^2 + 2^2} \cos \theta
\]

\( \Rightarrow \theta = 90^\circ \). That is, they are perpendicular. The shortest distance is produced by the projection.

- Min \( f(x, y) \) s.t. \( g_1 = g_2 = 0 \) \( \Rightarrow F = f + \lambda_1 g_1 + \lambda_2 g_2 \). Solve \( F_x = F_y = F_{\lambda_1} = F_{\lambda_2} = 0 \)
8.6 Total Differentials

- Let \( z = f(x, y) \) be a differentiable function
  
  1. Differentials of independent variables \( x \) and \( y \) are
     \[ dx = \Delta x \quad \text{and} \quad dy = \Delta y \]
  
  2. Differential of dependent variable \( z \) is
     \[ dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \]

- Total differential \( dz \): An approximation of the exact change \( \Delta z \)

\[
\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \approx dz
\]

provided \( \Delta x = dx \) and \( \Delta y = dy \) are sufficiently small.

- Example 1: Let \( z = 2x^2y + y^3 \).
Differential
\[dz = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = 4xy \, dx + (2x^2 + 3y^2) \, dy\]

Find the actual change in \(z\) when \(x\) changes from 1 to 1.01 and \(y\) changes from 2 to 1.98.

\[\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \approx -0.1980\]

Approximate change in \(z\): Here,
\[dx = 1.01 - 1, dy = 1.98 - 2 = -0.02\]

\[dz = 4(1)(2)(0.01) + [2(1)+3(4)](-0.02) = -0.20 \approx \Delta z\]

Example 4: Find the maximum percentage error in calculating the volume of a rectangular box if an error of at most 1% is made in measuring the length,
width, and height of the box.

- Let $x$, $y$, and $z$: length, width, and height.
  
  Volume $V = xyz$

- Suppose the true dimensions are $a$, $b$, and $c$ units.
  
  $|\Delta x| = |x-a| \leq 0.01a$, $|\Delta y| = |y-b| \leq 0.01b$, $|\Delta z| \leq 0.01c$

- Maximum error in calculating the volume

  \[
  \Delta V \approx |dV| = \left| \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right|_{x=a, y=b, z=c}
  \]

  \[
  = |bcdx + acdy + abdz|
  \]

  \[
  \leq bc(0.01a) + ac(0.01b) + ab(0.01c) = 0.03abc
  \]

- Max. percentage error

  \[
  \frac{|\Delta V|}{V}_{(a, b, c)} \approx \frac{(0.03)abc}{abc} = 3\%
  \]
8.7 Double Integrals

- Suppose $f(x, y)$ is a continuous function of two variables defined over a region $R$.

- Assume that $R$ is a rectangular region in the plane (Figure 8.29).

- A partition in the two-dimensional case is a rectangular grid composed of $mn$ rectangles, each of length $h$ and width $k$, where

  \[ h = \frac{b - a}{m}, \quad k = \frac{d - c}{n} \]

A sample grid with $m = 5$ and $n = 4$ is shown in Figure 8.30.
• Reimann sum of \( f(x, y) \) over the region \( R \) is

\[
S(m, n) = f(x_1, y_1)hk + \cdots + f(x_{mn}, y_{mn})hk
\]

• If the limit of \( S(m, n) \) exists as both \( m \) and \( n \) tend to infinity, we call this limit the value of the double integral of \( f(x, y) \) over the region \( R \) and denote it by

\[
\int \int_{R} f(x, y)dA
\]

• If \( f(x, y) \) is a nonnegative function, then it is the volume of the solid under the graph of \( f(x, y) \).
Evaluating a Double Integral Over a Rectangular Region

- Fubini’s Theorem: Let \( R \) be the rectangle defined by the inequalities \( a \leq x \leq b \) and \( c \leq y \leq d \) (Figure 8.30). Then,

\[
\int \int _{R} f(x, y) dA = \int _{c}^{d} \left[ \int _{a}^{b} f(x, y) dx \right] dy
\]

where the iterated integrals on the right-hand side are evaluated as follows.

- First compute the integral \( \int _{a}^{b} f(x, y) dx \) by treating \( y \) as if it were a constant and integrating the resulting function of \( x \) with respect to \( x \). Thus, \( \int _{a}^{b} f(x, y) dx = g(y) \) for some function \( g \).
– Then integrate $\int_c^d g(y)dy$

- Example 1: Evaluate $\iint_{R} f(x, y)dA$, where $f(x, y) = x + 2y$ and $R$ is the rectangle defined by $1 \leq x \leq 4$ and $1 \leq y \leq 2$.

  - (1) Evaluate $x$ first:
    $\iint_{R} f(x, y)dA = \int_1^2 \left[ \int_1^4 (x + 2y)dx \right] dy$
\[ \Rightarrow \int_{1}^{4} (x + 2y)\,dx = \frac{1}{2}x^2 + 2xy \bigg|_{x=1}^{x=4} = \frac{15}{2} + 6y \]

\[ \Rightarrow \int \int_{R} f(x, y)\,dA = \int_{1}^{2} \left( \frac{15}{2} + 6y \right)\,dy = 16\frac{1}{2} \]

- (2) Evaluate \( y \) first:

\[ \int \int_{R} f(x, y)\,dA = \int_{1}^{4} \left[ \int_{1}^{2} (x + 2y)\,dy \right] \,dx \]

\[ \Rightarrow \int_{1}^{2} (x + 2y)\,dy = xy + y^2 \bigg|_{y=2}^{y=1} = x + 3 \]

\[ \Rightarrow \int \int_{R} f(x, y)\,dA = \int_{1}^{4} (x + 3)\,dx = \frac{x^2}{2} + 3x \bigg|_{1}^{4} = 16\frac{1}{2} \]
Evaluating a Double Integral Over a Plane Region

• Theorem 1: (a) Suppose $g_1(x)$ and $g_2(x)$ are continuous functions on $[a, b]$ and the region $R = \{(x, y)|g_1(x) \leq y \leq g_2(x); a \leq x \leq b\}$ (Figure 8.32a). Then,

$$\int \int_R f(x, y) dA = \int_a^b \left[ \int_{g_1(x)}^{g_2(x)} f(x, y) dy \right] dx$$

• Theorem 1: (b) Suppose $h_1(x)$ and $h_2(x)$ are continuous functions on $[c, d]$ and the region $R = \{(x, y)|h_1(y) \leq x \leq h_2(y); c \leq y \leq d\}$ (Figure 8.32b). Then,

$$\int \int_R f(x, y) dA = \int_c^d \left[ \int_{h_1(y)}^{h_2(y)} f(x, y) dx \right] dy$$
Example 2: Evaluate $\int \int_R f(x, y) dA$, where $f(x, y) = x^2 + y^2$ and $R$ is the region bounded by the graphs of $g_1(x) = x$ and $g_2(x) = 2x$ for $0 \leq x \leq 2$. (Figure 8.33)

\[
\int \int_R f(x, y) dA = \int_0^2 \left[ \int_x^{2x} (x^2 + y^2) dy \right] dx
\]

\[
= \int_0^2 \left[ \left( x^2 y + \frac{1}{3} y^3 \right) \bigg|_x^{2x} \right] dx
\]

\[
= \int_0^2 \left[ \left( 2x^3 + \frac{8}{3} x^3 \right) - \left( x^3 + \frac{1}{3} x^3 \right) \right] dx
\]

\[
= \int_0^2 \left[ \frac{10}{3} x^3 \right] dx = \frac{5}{6} x^4 \bigg|_0^2 = \frac{40}{3}
\]

- Notice the sequence of integrations.
– Not the following

\[
\int_x^{2x} \left[ \int_0^2 (x^2 + y^2) \, dx \right] \, dy \\
= \int_x^{2x} \left[ \left( \frac{x^3}{3} + y^2 x \right) \bigg|_0^2 \right] \, dy \\
= \int_x^{2x} \frac{8}{3} + 2y^2 \, dy \\
= \left( \frac{8y}{3} + \frac{2y^3}{3} \right) \bigg|_x^{2x} = \frac{8x}{3} + \frac{14x^3}{3}
\]

* Why: Integration over a constant planar region must be a constant.

* The region?
Example 3: Evaluate $\int \int_{R} f(x, y) \, dA$, where $f(x, y) = xe^y$ and $R$ is the plane region bounded by the graphs of $y = x^2$ and $y = x$. (Figure 8.34)

- The point of intersection is found by solving the equation $x^2 = x$, giving $x = 0$ and $x = 1$.
- Then,

$$\int \int_{R} f(x, y) \, dA = \int_{0}^{1} \left[ \int_{x^2}^{x} xe^y \, dy \right] \, dx$$

$$= \int_{0}^{1} \left[ xe^y \right]_{x^2}^{x} \, dx = \int_{0}^{1} \left( xe^x - xe^{x^2} \right) \, dx$$

$$= \left[ (x - 1)e^x - \frac{1}{2} e^{x^2} \right]_{0}^{1} = \frac{1}{2} (3 - e)$$
Another method: $\int_0^1 \left[ \int_y^\sqrt{y} xe^y \, dx \right] \, dy$

Example 4: Evaluate $\int \int_R xe^{y^2} \, dA$, where $R$ is the plane region bounded by the $y$-axis, the horizontal line $y = 4$, and the graph of $y = x^2$. (Figure 8.35)

- Point of intersection of $y = 4$ and $y = x^2$ is $(2, 4)$.

$$\int \int_R xe^{y^2} \, dA = \int_0^2 \left[ \int_{x^2}^4 xe^{y^2} \, dy \right] \, dx$$

- Now evaluation of the integral

$$\int_{x^2}^4 xe^{y^2} \, dy = x \int_{x^2}^4 e^{y^2} \, dy$$

calls for finding the antiderivative of the integrand $e^{y^2}$ in terms of elementary functions. Not feasible
Since the equation $y = x^2$ is equivalent to the equation $x = \sqrt{y}$, we may write $x = h_1(y) = 0$ and $x = h_2(y) = \sqrt{y}$,

\[
\int \int_{R} x e^{y^2} \, dA = \int_{0}^{4} \left[ \int_{0}^{\sqrt{y}} x e^{y^2} \, dx \right] \, dy = \int_{0}^{4} \left[ \frac{1}{2} x^2 e^{y^2} \bigg|_{0}^{\sqrt{y}} \right] \, dy
\]

\[
= \int_{0}^{4} \frac{1}{2} y e^{y^2} \, dy = \frac{1}{4} e^{y^2} \bigg|_{0}^{4} = \frac{1}{4} (e^{16} - 1)
\]
• Compute $\int_0^1 \int_0^1 e^{\max(x^2, y^2)} \, dx \, dy$ where $\max(x^2, y^2)$ means the larger of the numbers $x^2$ and $y^2$

\[
\int_0^1 \int_0^1 e^{\max(x^2, y^2)} \, dx \, dy = \int_0^1 \int_0^x e^{x^2} \, dy \, dx + \int_0^1 \int_0^y e^{y^2} \, dx \, dy
\]
\[
= \int_0^1 xe^{x^2} \, dx + \int_0^1 ye^{y^2} \, dy = \left. \frac{e^{x^2}}{2} \right|_0^1 + \left. \frac{e^{y^2}}{2} \right|_0^1
\]
\[
= e - 1
\]
8.8 Applications of Double Integrals

- Example 1: Find the volume of the solid bounded above by the plane \( z = f(x, y) = y \) and below by the plane region \( R \) defined by \( y = \sqrt{1 - x^2}, 0 \leq x \leq 1 \).
  - The region \( R \) is sketched in Figure 8.36.
  - Observe that \( f(x, y) = y \geq 0 \) for \( y \in R \).

Therefore, the required volume is

\[
\int \int_{R} y \, dA = \int_{0}^{1} \left[ \int_{0}^{\sqrt{1-x^2}} y \, dy \right] \, dx = \int_{0}^{1} \left[ \frac{1}{2} y^2 \right]^{\sqrt{1-x^2}}_{0} \, dx
\]

\[
= \int_{0}^{1} \frac{1}{2} (1 - x^2) \, dx = \frac{1}{2} \left( x - \frac{1}{3} x^3 \right) \bigg|_{0}^{1} = \frac{1}{3}
\]
Example 2: The population density of a certain city is

\[ f(x, y) = 10,000e^{-0.2|x| - 0.1|y|} \]

where the origin \((0, 0)\) gives the location of the city hall. What is the population inside the rectangular area described by

\[ R = \{(x, y) | -10 \leq x \leq 10; -5 \leq y \leq 5\} \]

if \(x\) and \(y\) are in miles?

- By symmetry, it suffices to compute the population in the first quadrant. (Why?)
- In this quadrant, \(f(x, y) = 10,000e^{-0.2x - 0.1y}\)
The population in $R$ is

$$\int \int_R f(x, y)dA = 4 \int_{0}^{10} \left[ \int_{0}^{5} 10,000 e^{-0.2x} e^{-0.1y} dy \right] dx$$

$$= 4 \int_{0}^{10} \left[ -100,000 e^{-0.2x} e^{-0.1y} \right]_{0}^{5} dx$$

$$= 400,000(1 - e^{-0.5}) \int_{0}^{10} e^{-0.2x} dx$$

$$= 2,000,000(1 - e^{-0.5})(1 - e^{-2}) \approx 680,438$$

Consider the improper double integral $\int \int_D f(x, y)dA$ of the continuous function $f$ of two variables defined over the plane region

$$D = \{(x, y)|0 \leq x < \infty; 0 \leq y < \infty\}$$
Under the definition of improper integrals of functions of one variable, it makes sense to define

\[
\int \int_D f(x, y) \, dA = \lim_{N \to \infty} \int_0^N \left[ \lim_{M \to \infty} \int_0^M f(x, y) \, dx \right] \, dy
\]

\[
= \lim_{M \to \infty} \int_0^M \left[ \lim_{N \to \infty} \int_0^N f(x, y) \, dy \right] \, dx
\]

provided the limits exist.

- Average value of \( f(x, y) \) over the region \( R \)

\[
\frac{f(x_1, y_1) + \cdots + f(x_{mn}, y_{mn})}{mn}
\]
\[
\frac{hk}{h} n \left[ f(x_1, y_1) + \cdots + f(x_{mn}, y_{mn}) \right]
\]

\[
= \frac{1}{(mn)hk} \left[ f(x_1, y_1) + \cdots + f(x_{mn}, y_{mn}) \right] hk
\]

Taking the limit as \( m \) and \( n \) both tend to infinity, we obtain the following formula for the average value of \( f \) over \( R \)

\[
\frac{\int \int_{R} f(x, y) \, dA}{\int \int_{R} \, dA}
\]

**Example 3:** Find the average value of the function \( f(x, y) = xy \) over the plane region defined by \( y = e^x, \, 0 \leq x \leq 1 \).
– The region $R$ is shown in Figure 8.40.

– Method 1: The area of the region $R$ is

$$\int_{0}^{1} \left[ \int_{0}^{e^{x}} dy \right] dx = \int_{0}^{1} [y]_{0}^{e^{x}} dx = \int_{0}^{1} e^{x} dx$$

$$= e^{x}|_{0}^{1} = e - 1$$

– Method 2:

$$\int_{1}^{e} \left[ \int_{\ln y}^{1} dx \right] dy + 1 = \int_{1}^{e} [1 - \ln y] dy + 1$$

$$= [y - (y \ln y - y)]|_{1}^{e} + 1$$

$$= (2e - e \ln e) - (2 - 0) + 1$$
= \ e - 1

– Integration of \ f \ over \ R \ is:
\[ \int \int_R f(x, y) \, dA \ = \ \int_0^1 \left[ \int_0^{e^x} xy \, dy \right] \, dx = \int_0^1 \left[ \frac{1}{2} xy^2 \right]_{e^x}^{0} \, dx \]
\[ = \ \int_0^1 \frac{1}{2} x e^{2x} \, dx = \frac{1}{4} xe^{2x} - \frac{1}{8} e^{2x} \Big|_0^1 \]
\[ = \ \frac{1}{8} (e^2 + 1) \]

– Average value
\[ \frac{\int \int_R f(x, y) \, dA}{\int \int_R dA} = \frac{\frac{1}{8} (e^2 + 1)}{e - 1} = \frac{e^2 + 1}{8(e - 1)} \]
8.9 Double Integrals in Polar Coordinates

- Reference: Smith and Minton, Calculus, Section 13.3
- Motivating example: Evaluate \( \int \int_R (x^2 + y^2 + 3) \, dA \) where \( R \) is the circle of radius 2, centered at the origin. (Figure 13.26)
  - Then
  \[
  \int \int_R (x^2 + y^2 + 3) \, dA \\
  = \int_{-2}^{2} \int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2 + y^2 + 3) \, dy \, dx \\
  = 2 \int_{-2}^{2} \left[ (x^2 + 3) \sqrt{4 - x^2} + \frac{1}{3} (4 - x^2)^{3/2} \right] \, dx
  \]
  - Not pleasant form and difficult to integrate
• Suppose the region $R$ can be written in the form

$$R = \{(r, \theta) | \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)\}$$

- Transformation: $x = r \cos \theta, y = r \sin \theta$. And $r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} \frac{y}{x}$.

- Use a partition consisting of a number of concentric circular arcs (of the form $r = \text{constant}$) and rays (of the form $\theta = \text{constant}$) (Figure 13.27b)

- Area $\triangle A$ of the elementary polar region (Figure 13.27c). Let $\bar{r} = \frac{1}{2}(r_1 + r_2)$. Then
\[ \Delta A = \frac{1}{2} \Delta \theta r_2^2 - \frac{1}{2} \Delta \theta r_1^2 \]
\[ = \frac{1}{2} (r_2^2 - r_1^2) \Delta \theta \]
\[ = \frac{1}{2} (r_2 + r_1)(r_2 - r_1) \Delta \theta \]
\[ = \bar{r} \Delta r \Delta \theta \]

- Consider the volume under the function \( f \) over \( R \). The volume \( V_i \) lying beneath \( z = f(r, \theta) \) and above the \( i \)th elementary polar region is then approximately the volume of the cylinder:
\[ V_i \approx f(r_i, \theta_i) \Delta A_i = f(r_i, \theta_i) r_i \Delta r_i \Delta \theta_i \]

- The volume is given by (Fubini’s Theorem)

\[ V = \lim_{n \to \infty} \sum_{i=1}^{n} f(r_i, \theta_i) r_i \Delta r_i \Delta \theta_i = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta) r dr d\theta \]

- Example 3.2: Motivating example

\[ \int \int_{R} (x^2 + y^2 + 3) dA = \int_{0}^{2\pi} \int_{0}^{2} (r^2 + 3)r dr d\theta \]

\[ = \int_{0}^{2\pi} \left[ \int_{0}^{2} (r^3 + 3r) dr \right] d\theta = \int_{0}^{2\pi} \left( \frac{r^4}{4} + \frac{3}{2} r^2 \right) \bigg|_{0}^{2} d\theta \]

\[ = 10 \int_{0}^{2\pi} d\theta = 20\pi \]
Example 3.4

\[ A = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} x^2 (x^2 + y^2)^2 \, dy \, dx \]

- Evaluating this integral in rectangular coordinates is nearly hopeless.
- The integration region is a semicircle. (Figure 13.31) Then

\[
A = \int \int_{R} (r \cos \theta)^2 (r^2)^2 r \, dr \, d\theta
\]

\[
= \int_{0}^{\pi} \int_{0}^{1} r^7 \cos^2 \theta \, dr \, d\theta = \int_{0}^{\pi} \left. \frac{r^8}{8} \right|_{0}^{1} \cos^2 \theta \, d\theta
\]

\[
= \frac{1}{8} \int_{0}^{\pi} \frac{1}{2} (1 + \cos 2\theta) \, d\theta = \frac{1}{16} \left( \theta + \frac{1}{2} \sin 2\theta \right) \bigg|_{0}^{\pi} = \frac{\pi}{16}
\]
• Example: A ball $x^2 + y^2 + z^2 = c^2$ with radius $c$. The height is $z = \sqrt{c^2 - x^2 - y^2}$. Compute the volume of the upper semi-ball and multiply by 2:

$$V = 2 \int \int \int_{x^2+y^2 \leq c^2} \sqrt{c^2 - x^2 - y^2} \, dx \, dy$$

$$= 2 \int_{0}^{2\pi} \int_{0}^{c} \sqrt{c^2 - r^2} r \, dr \, d\theta = 2 \int_{0}^{2\pi} \left[ -\frac{1}{3} (c^2 - r^2)^{3/2} \right]_{0}^{c} \, d\theta$$

$$= 2 \int_{0}^{2\pi} \frac{1}{3} c^3 \, d\theta = \frac{4\pi c^3}{3}$$

• Exercise: $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2} \, dy \, dx = \frac{16\pi}{3}$

• Exercise: $\int_{0}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{-x^2-y^2} \, dy \, dx = \frac{\pi}{2} (1 - e^{-4})$

• Region:
8.10 Change of Variables in Multiple Integrals

- Reference: Smith and Minton, Calculus, Section 13.8

- Theorem 8.1 Change of variables in double integrals: Suppose that the region $S$ in the $uv$-plane is mapped onto the region $R$ in the $xy$-plane by the one-to-one transformation $T$ defined by $x = g(u, v)$ and $y = h(u, v)$, where $g$ and $h$ have continuous first partial derivatives on $S$. If $f$ is continuous on $R$ and the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ is nonzero on $S$, then

$$\int \int_R f(x, y) \, dA = \int \int_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv,$$
∂(x, y) ∂(u, v) = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}

- Example 8.3 Changing variables to polar coordinates
  - Consider the transformation from the \( r\theta \)-plane to the \( xy \)-plane defined by \( x = r \cos \theta \) and \( y = r \sin \theta \)
  - The Jacobian

\[
\frac{\partial (x, y)}{\partial (r, \theta)} = \left| \begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{array} \right| = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r
\]
  - By Theorem 8.1, we have
\[ \int \int_{R} f(x, y) dA = \int \int_{S} f(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| drd\theta \]

\[ = \int \int_{S} f(r \cos \theta, r \sin \theta) r drd\theta \]

- Example 8.4 (Changing variables to transform a region) Evaluate \( \int \int_{R} (x^2 + 2xy) dA \), where \( R \) is the region bounded by the lines \( y = 2x + 3 \), \( y = 2x + 1 \), \( y = 5 - x \), and \( y = 2 - x \).

  - Difficulty: The region of integration requires us to break the integral into 3 pieces. (Figure 13.74)

  - The region \( R \) is a parallelogram in the \( xy \)-plane. This suggests the change of variables
\[ u = y - 2x, \ v = y + x \Rightarrow x = \frac{1}{3}(v - u), \ y = \frac{1}{3}(2v + u) \]

- The new lines forming the boundaries of \( R \) are the lines \( u = 3, u = 1, v = 5, \) and \( v = 2, \) respectively.

- The Jacobian of this transformation is

\[
\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \begin{vmatrix}
-\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{2}{3}
\end{vmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3}
\]

\[
\Rightarrow \quad \int \int_{R} (x^2 + 2xy)dA = \int \int_{S} \left[ \frac{1}{9}(v - u)^2 + \frac{2}{9}(v - u)(2v + u) \right] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv
\]
\[
\begin{align*}
&= \frac{1}{27} \int_2^5 \int_1^3 \left[ (v - u)^2 + 2(2v^2 - uv - u^2) \right] \, dudv = \frac{196}{27} \\
&- \text{Area of the parallelogram bounded by the lines} \\
\int \int_R \, dA = \frac{1}{3} \int_2^5 \int_1^3 \, dudv = \frac{(5 - 2)(3 - 1)}{3} = 2 \\
&- \text{Example 8.5 (A change of variables required to find an antiderivative)} \text{ Evaluate } \int \int_R \frac{e^{(x-y)}}{x+y} \, dA, \text{ where } R \text{ is the rectangle bounded by the lines } y = x, y = x + 5, y = 2 - x, \text{ and } y = 4 - x. \text{ (Figure 13.75 a)} \\
&- \text{ The integration region is a rectangle, but we do not know an antiderivative for the integrand.} \\
&- \text{ Change of variables}
\end{align*}
\]
\[ u = x - y, v = x + y \Rightarrow x = \frac{1}{2}(u + v), y = \frac{1}{2}(v - u) \]

\[ \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} \]

\[ = \frac{1}{4} - \left( -\frac{1}{4} \right) = \frac{1}{2} \]

- The region \( S \) in the \( uv \)-plane corresponding to the region \( R \) in the \( xy \)-plane is the rectangle

\[ S = \{ (u, v) | -5 \leq u \leq 0, \quad 2 \leq v \leq 4 \} \]. Then

\[ \int \int_{R} e^{x-y} \frac{\partial(x, y)}{\partial(u, v)} \, dA = \int \int_{S} e^{u} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dudv \]

\[ = \frac{1}{2} \int_{2}^{4} \int_{-5}^{0} e^{u} \, dudv = \frac{1}{2} \int_{2}^{4} e^{u} \left|_{u=0}^{u=4} \right| \, dv \]
\[
\frac{1}{2} (e^0 - e^{-5}) \int_2^4 \frac{1}{v} dv = \frac{1}{2} (1 - e^{-5}) \ln |v| \bigg|_2^4 \\
= \frac{1}{2} (1 - e^{-5})(\ln 4 - \ln 2)
\]

- The volume of the parallelepiped bounded by \( R \), where \( R \) is the region enclosed by \( x + y + z = 1 \), \( x + y + z = 2 \), \( x + 2y = 0 \), \( x + 2y = 1 \), \( y + z = 2 \), and \( y + z = 4 \).

- Let \( u = x + y + z \), \( v = x + 2y \), and \( w = y + z \), then \( x = u - w \), \( y = \frac{1}{2}(-u + v + w) \), and \( z = \frac{1}{2}(u - v + w) \). Then Jacobian:
\[
\frac{\partial(x, y, z)}{\partial(u, v, w)} = \left| \begin{array}{ccc}
1 & 0 & -1 \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{array} \right| = \frac{1}{2}
\]

- Volume:

\[
\int \int \int_R \, dxdydz = \frac{1}{2} \int_2^4 \int_0^1 \int_1^2 \, dudvdw = \frac{2 \times 1 \times 1}{2} = 1
\]
Spherical Coordinates and Triple Integrals

- Spherical coordinates \((\rho, \phi, \theta)\) where
  \[ \rho = \sqrt{x^2 + y^2 + z^2}. \]

- From Figure 13.59, we have \(\rho \geq 0\) and \(0 \leq \phi \leq \pi\).

- Relate rectangular and spherical coordinates

  \[
  x = ||QP|| \cos \theta = \rho \sin \phi \cos \theta,
  \]
  \[
  y = ||QP|| \sin \theta = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi.
  \]

- The Jacobian of the above transformation
\[
\frac{\partial (x, y, z)}{\partial (\rho, \phi, \theta)} = \left| \begin{array}{ccc}
\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\
\frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}
\end{array} \right|
\]

\[
= \left| \begin{array}{ccc}
\sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\
\sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\
\cos \phi & -\rho \sin \phi & 0
\end{array} \right|
\]

\[
= \rho^2 \cos^2 \phi \cos^2 \theta \sin \phi + \rho^2 \sin^3 \phi \sin^2 \theta + \rho^2 \sin^3 \phi \cos^2 \theta
\]

\[
+ \rho^2 \cos^2 \phi \sin^2 \theta \sin \phi
\]

\[
= \rho^2 \cos^2 \phi \sin \phi + \rho^2 \sin^3 \phi = \rho^2 \sin \phi
\]
• Volume of a ball with radius $r$:

• Exercise: $\int \int \int_{Q} (x^2 + y^2) \, dV = \frac{32\pi}{3}$, where $Q$ is bounded by $z = 4 - x^2 - y^2$ and the $xy$-plane.